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Upper and lower bounds for the bounce action

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Abstract. We show how to bound the WKB exponent in field theory from above and below. The results follow from an inf-max characterisation of the Euclidian action of the bounce solution. For the lower bounds our method involves only some simple inequalities while for the upper bounds it leads to a finite dimensional extremising problem. We apply this method to approximate the WKB exponent for the effective potential in 1-loop approximation which is used in the inflationary cosmological models.

1. Introduction

During recent years we have seen an increasing interrelation between cosmology and particle physics. The recent and exciting scenarios for the very early universe are not thinkable without this interplay. For example, all the inflationary models presented up to now (Guth 1981, Linde 1982, Albrecht and Steinhardt 1982) need a scalar field as order parameter. Its self interaction causes the long de Sitter period in the history of the universe and delays the expected phase transition. This strongly first-order phase transition is associated with the spontaneous symmetry breakdown of a grand unified gauge theory.

In spite of intensive study over the last few years (see e.g. Gunton and Droz 1983) the theory for the condensation of the metastable symmetric state into the stable asymmetric state is imperfectly understood. However, there has been some progress in understanding this phenomenon on a semiclassical level (Langer 1967, Coleman 1977, Callan and Coleman 1977) using multi-instanton techniques. In the 1-loop approximation the decay rate per volume and time of the metastable state has been computed to be

$$\Gamma/\mathrm{Vol} \simeq C \exp(-S[\phi_{\mathrm{B}}]/\hbar) \tag{1}$$

where S denotes the Euclidian action

$$\begin{aligned} S[\phi] &= \int L_{\mathrm{eff}}(\phi)\,\mathrm{d}^d x = \tfrac{1}{2}\|\nabla\phi\|_2^2 + \int \mathrm{d}^d x V(\phi) \\ &= T[\phi] + V[\phi] \end{aligned} \tag{2}$$

and C is the 1-loop correction to the classical exponential factor. The effective interaction Lagrangian is formally given by integrating out the gauge- and fermionic fields, i.e.

$$\exp(-L_{\mathrm{eff}}(\phi)) = Z^{-1} \int \exp(-L(\phi, A, \psi, \psi^*))\,\mathrm{d}\psi\,\mathrm{d}\psi^*\,\mathrm{d}A. \tag{3}$$

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For our purpose $V(\phi)$ is assumed to have somewhere a *local* minimum. We normalise V such that $\phi=0$ is this local minimum and $V(0)=0$ as shown in figure 1. As models for $V(\phi)$ we choose

$$V(\phi)=\tfrac{1}{2}m^2\phi^2-(c/q)\phi^q+(a/p)\phi^p \tag{4}$$

with $2<q<p\leq d_c$ and

$$V(\phi)=(2A-B)\sigma^2\phi^2+A\phi^4+B\phi^4\ln(\phi^2/\sigma^2) \tag{5}$$

with $\frac{1}{2}<Q=A/B\leq 1$. The critical dimension $d_c=2d/(d-2)$ is the largest exponent in (4) up to which the corresponding theory is renormalisable. Both potentials have been used to force a first-order phase transition in the very early universe.

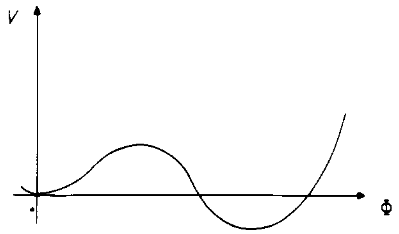

Figure 1.

The function ϕ_B is a solution of the classical Euclidian equation of motion

$$\Delta\phi_B=V'(\phi). \tag{6}$$

Furthermore ϕ_B must satisfy

$$\lim_{|x|\to\infty}\phi_B(x)=0 \tag{6'}$$

and must have minimal Euclidian action.

A lot is known about the existence and properties of the bounce solution ϕ_B, e.g. it is spherically symmetric and monotonically and exponentially decaying for a large class of potentials with positive mass (Strauss 1977, Coleman *et al* 1978, see, e.g., Blanchard and Brüning 1982). In this paper we use a variational characterisation for the action $S_B=S[\phi_B]$. The advantage is that without explicit knowledge of $\phi_B(x)$ we are able to find bounds for the WKB exponent in (1). In the derivation of lower bounds we only need some additional simple inequalities, e.g. the Hölder–Young (HY), Sobolev and Jensen inequality.

On the other hand the variational method naturally delivers upper bounds for S_B. Irrespective of the application in mind the invented method is interesting in its own right. It is by far the most efficient algorithm for computing $S[\phi_B]$ which is known to us.

2. Inf–max principle for the bounce action

It has been proved (Wipf 1985) that the action S_B can be calculated with the help of a variational principle. For its formulation we introduce the set

$$\Omega=\{\phi\in H|\rho_d V[\phi]<0\} \tag{7}$$

where $\rho_d = \text{sgn}(d_c)$ and H is some subspace of H^1_r which depends on the potential. With it the principle reads

$$S_B = \rho_d \inf_{\phi_1 \in \Omega} \max_{t \in [0,1]} \rho_d S[\phi_t] \tag{8}$$

and is valid in all dimensions d except $d=2$. Here ϕ_t denotes the function $\phi_t(x) = \phi(x/t)$. Note that in one dimension the inf-max principle becomes a sup-min principle for $S[\phi_t]$.

The equality (8) especially holds for the admissible potentials in the sense of Coleman *et al* (CGM) and hence is applicable to a large variety of Higgs models.

For functionals of the form (2) the maximum can be computed explicitly. Since $T[\phi_t] = t^{d-2} T[\phi_1]$ and $V[\phi_t] = t^d V[\phi_1]$ we obtain

$$S_B = \rho_d / d \inf_{\phi_1 \in \Omega} \rho_d (2T)^{d/2} (-d_c V)^{-d/d_c}. \tag{9}$$

It is convenient to use a dimensionless field and potential. For that purpose we introduce a typical field amplitude σ, i.e. a point of V to which the field tunnels. By setting

$$U[\psi] = 2(m\sigma)^{-2} V[\sigma\psi]. \tag{10}$$

The action S_B as functional of ψ becomes

$$S_B = (\rho_d / d) m^{2-d} \sigma^2 \inf_{\Omega} \rho_d (2T)^{d/2} (-d_c/2U)^{-d/d_c}. \tag{9$'$}$$

This characterisation of S_B will serve as the starting point for the following considerations.

3. Lower bounds

In proving lower bounds for $S[\phi_B]$ we apply the following strategy. First we deduce an inequality of the form

$$2T[\psi] \geq F_d(\|\psi\|_p, \|\psi\|_2) = F_d[p, \psi] \tag{11}$$

for some $2 < p \leq d_c$. Next we derive an inequality

$$U[\psi] \geq G(\|\psi\|_p, \|\psi\|_2) = G[p, \psi] \tag{12}$$

such that

$$H_d(p, \eta) = F_d^{d/2} (-d_c/2G)^{-d/d_c} \tag{13}$$

is a function only of $\eta = \|\psi\|_p^p / \|\psi\|_2^2$. Because of (9$'$) we get the lower bound

$$S_B \geq S_L = (m^{2-d} \sigma^2 / d) \rho_d \min_{\eta \in \Delta} \rho_d H_d(p, \eta) \tag{14}$$

where $\Delta = \{\eta | \rho_d G < 0\}$. It requires only some simple inequalities to find F_d. By combining the HY inequality of the form

$$\|\psi\|_{d_c}^{d_c} \geq \|\psi\|_2^2 \cdot \eta^{(d_c-2)/(p-2)}$$

which holds for $2 < p \leq d_c$, with the Sobolev inequality

$$\|\nabla\psi\|_2 \geq c_d \|\psi\|_{d_c} \tag{15}$$

valid in more than two dimensions, we obtain the desired function

$$F_d[p,\psi]=c_d^2\eta^{4/d(p-2)}\|\psi\|_2^{4/d_c}. \tag{16}$$

For $d=1$ we apply the inequality

$$\frac{|(\psi,v(x)\psi)|}{\|\psi\|_2^2}\leqslant\int v(x)\,\mathrm{d}x\frac{\|\nabla\psi\|_2}{\|\psi\|_2}$$

due to Faris (1978). Setting $v(x)=|\psi(x)|^{p-2}$ and applying once more the HY inequality we find

$$\|\nabla\psi\|_2^2\geqslant\frac{\|\psi\|_p^p}{\|\psi\|_2^2\|\psi\|_{p-2}^{4p-4}}\geqslant\eta^{4/(p-2)}\|\psi\|_2^{-2} \tag{17}$$

which delivers us exactly $F_1[p,\psi]$ as defined in (16). With (13), (14) and (16) we finally end up with

$$S_B\geqslant S_L=\rho_d\beta_d m^{2-d}\sigma^2\min_{\Delta}\rho_d\eta^{2/(p-2)}(2\|\psi\|_2^2/-d_cG)^{d/d_c}. \tag{18}$$

Using the sharpest constants in (15) and the result (17)

$$\begin{aligned}&\beta_1=1 && \text{for } d=1\\ &\beta_d=(1/d)[\pi d(d-2)]^{d/2}\Gamma(d/2)/\Gamma(d) && \text{for } d>2.\end{aligned} \tag{19}$$

The inequality (18) holds for $d>2$ and $2<p\leqslant d_c$ or for $d=1$ and $p>2$. Clearly the functional $G[p,\psi]$ depends on the chosen potential. We now apply our result to the models (4) and (5).

3.1. *Application to the model (4)*

In order to shorten the computations we derive the bounds for $c>0$ and $a=0$. The more general case can be handled in exactly the same way by using the HY inequality a second time. Let $\sigma=(qm^2/2c)^{1/(q-2)}$ be the positive zero of V. We may choose for G in (11)

$$G_d[p,\psi]/\|\psi\|_2^2=U[\psi]/\|\psi\|_2^2=1-\eta$$

with $\eta=\|\psi\|_q^q/\|\psi\|_2^2$. The corresponding $H_d(q,\eta)$ as defined in (13) looks qualitatively as shown in figure 2. It attains its extremum at

$$\eta_0(d,q)=4/[2q-d(q-2)] \tag{20}$$

and therefore (18) yields

$$S_B\geqslant S_L=\beta_d m^{2-d}\sigma^2\eta_0^{2/(q-2)}[(d/4)(q-2)\eta_0]^{-d/d_c}. \tag{21}$$

It is worth recognising that $S_L\to\infty$ for $q\nearrow d_c$. This partly proves the well known theorem that

$$\Delta\phi=m^2\phi-c\phi^{q-1}$$

has no finite energy solution for $q\geqslant d_c$ (see, e.g., Blanchard and Brüning 1982). We will compare the bounds (21) with known exact results in the following section.

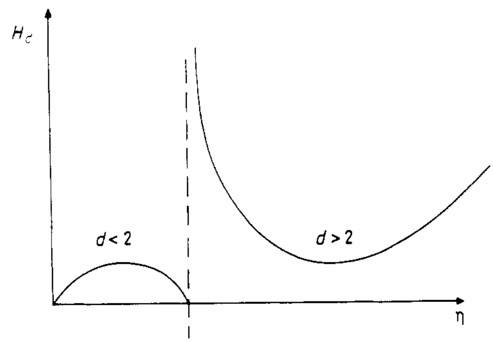

Figure 2.

3.2. Application to the model (5)

In order to find a bound for (12) we use the Jensen inequality

$$\int \mathrm{d}P(x)\, \mathrm{e}^{f(x)} \geqslant \exp\left(\int \mathrm{d}P(x) f(x)\right)$$

with $\mathrm{d}P(x) = \psi^4(x)/\|\psi\|_4^4$ and $f = -\ln \psi^2$ to obtain

$$L[\psi] = \int \frac{\psi^4}{\|\psi\|_4^4} \log \eta/\psi^2 \leqslant 0 \tag{22}$$

where $\eta = \|\psi\|_4^4/\|\psi\|_2^2$. With (5), (10) and (12) we may choose

$$(1-2Q)G[\psi]/\|\psi\|_2^2 = g(\eta) = 1-2Q+Q\eta-\eta\log\eta \tag{23}$$

which, together with (18) delivers for $d > 2$

$$S_{\mathrm{L}} = \beta_d B^{-d/d_c} \sigma^{4-d} \min_{\Delta} \eta\{d_c g(\eta)\}^{-d/d_c}. \tag{24}$$

The equation which extremises S_{L} with respect to η

$$(4-d)\eta\log\eta - \eta[(4-d)Q+(d-2)]+2(2Q-1)=0, \tag{25}$$

is analytically solvable only in four dimensions. With $B = (75\alpha/8)^2$, $\alpha \simeq \frac{1}{45}$ we find the lower bound

$$S_{\mathrm{L}}(d=4) = \tfrac{2}{3}(8\pi/75\alpha)^2[Q-1-\log(2Q-1)]^{-1}. \tag{26}$$

In other dimensions equation (25) must be solved numerically. We computed min $H_d(\eta)$ in three dimensions. The corresponding graph $S_{\mathrm{L}}(d=3, \eta)$ is shown in figure 4.

4. Upper bounds

Clearly any trial function in (9) gives an upper bound for the bounce action. The only problem arises in finding suitable trial functions to get sharp bounds. We will see that even very simple functions give astonishingly good results.

As first and natural variational parameters we choose the amplitude of ψ. Hence we replace ψ in (9′) by $y \cdot \chi$ and minimise with respect to the dimensionless parameter y.

We finally choose a family of normalised trial functions $\chi_\alpha(x)$ depending on s variational parameter $\{\alpha_1, \ldots, \alpha_s\}$. It is convenient to introduce

$$v(y, \alpha) = y^{-2} U[y \cdot \chi_\alpha], \qquad t(\alpha) = T[\chi_\alpha] \tag{27}$$

in terms of which the upper bound becomes

$$S_{\mathrm{B}} \leq S_{\mathrm{U}} = \frac{m^{2-d}\sigma^2}{d} \min_{\delta} y^2 (2t)^{d/2} (-d_{\mathrm{c}}/2 \cdot v)^{-d/d_{\mathrm{c}}} \tag{28}$$

where $\delta = \{(y, \alpha) \in R^{1+s} | y \cdot \chi_\alpha \in \Delta\}$. Sometimes it is possible to find the minimising y, which obeys

$$4v = (d-2) y \, \mathrm{d}v/\mathrm{d}y \tag{29}$$

explicitly.

(i) For the potential $V = \frac{1}{2} m^2 \phi^2 - (c/q)\phi^q$ this equation yields $y^{q-2} = \eta_0(q, d)/\eta$ and (28) becomes

$$S_{\mathrm{U}} = \rho_d S_{\mathrm{L}} \min_{\alpha} \rho_d (2t(\alpha)/F_d[q, \chi_\alpha])^{d/2} \tag{30}$$

where S_{L} is given in (21).

(ii) Equation (29) can be solved analytically for the effective potential (4) only in four dimensions. We find $y^2 = (2Q-1)/\eta$ and using $S_{\mathrm{L}}(d=4)$ in (26)

$$S_{\mathrm{U}}(d=4) = \rho_d S_{\mathrm{L}}(d=4) \min_{\alpha} \rho_d \left(\frac{2t(\alpha)}{F_4[4, \chi_\alpha]}\right)^2 \left(1 + \frac{L[\chi_\alpha]}{Q - 1 - \log(2Q-1)}\right)^{-1}. \tag{31}$$

It is worthwhile noticing that (30) and (31) together with (11) and (22) again give the lower bounds (21) and (26).

5. Numerical results

Depending on the value of $\varepsilon = -U$ (absolute minimum) we choose different trial functions. For $\varepsilon \simeq 1$ we use

$$\chi_\alpha(x) = \exp(-r^\alpha/2) \tag{32}$$

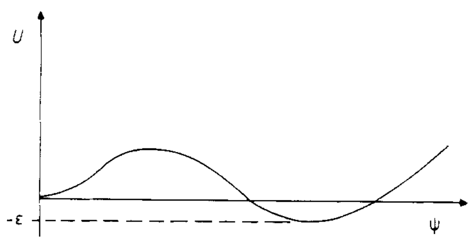

Figure 3.

and in the 'thin wall' region (Coleman 1977) $\varepsilon \ll 1$

$$\chi_\alpha(x)=\begin{cases}1 & \text{if } r\leqslant\alpha\\ \exp[\frac{1}{2}(r-\alpha)^2] & \text{if } r>\alpha.\end{cases} \tag{33}$$

This choice is suggested by the interpretation of the field equation (6) for a spherically symmetry field

$$\ddot{\phi}_{\mathrm{B}}+[(d-1)/r]\dot{\phi}_{\mathrm{B}}=V'(\phi_{\mathrm{B}})$$

as the equation of motion for a point particle in the potential $-V$ which suffers a friction force.

Next we compare the variational bounds with known exact results. In one dimension $S_{\mathrm{B}}=\int_0^\sigma (2V(x))^{1/2}\,\mathrm{d}x$ is explicitly known. For $V(\phi)=\frac{1}{2}m^2\phi^2(1-\phi/\sigma)$ we obtain $S_{\mathrm{B}}=0.5333\, m\sigma^2$. On the other hand, using (21) and (30) wherein we take the trial function (32) we find the bounds $S_{\mathrm{L}}=0.2862m\sigma^2$, and $S_{\mathrm{U}}=0.5346m\sigma^2$ respectively. Note that with only a 1-parametric trial function the relative error $(S_{\mathrm{U}}-S_{\mathrm{B}})/S_{\mathrm{B}}\simeq 0.002$ is astonishingly small.

For the potential $V(\phi)=\frac{1}{2}m^2\phi^2(1-\phi^2/\sigma^2)$ the bounce action in three dimensions is known to be $S_{\mathrm{B}}=18.90\sigma^2/m$ (Brezin and Parisi 1978). Again using the function (32) we obtain $S_{\mathrm{U}}=19.27\sigma^2/m$ or $(S_{\mathrm{U}}-S_{\mathrm{B}})/S_{\mathrm{B}}\simeq 0.019$. These two (and some others) examples show that we developed a very simple and accurate approximation scheme for computing the minimal action S_{B}.

With the cosmological application in mind we computed S_{U} and S_{L} for the effective potential (5) in three and four dimensions. The calculation of $S_{\mathrm{U}}(Q)$ has been carried out with the two trial functions (32) and (33). Figures 4 and 5 show for each $Q\in(\frac{1}{2},1)$ the lower of these two values.

We also calculated and plotted the thin wall approximation S_{tw} (Coleman 1977, Wipf 1985) to S_{B}. For $2\varepsilon=(1-Q)/(2Q-1)\searrow 0$ or $Q\nearrow 1$ where this approximation is valid, S_{U} and S_{tw} approach each other.

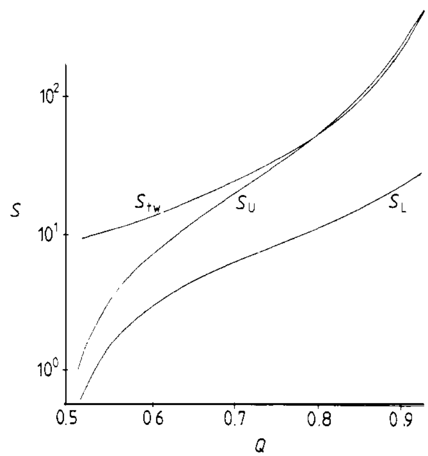

Figure 4. Plots of the lower bound S_{L}, upper bound S_{U} and thin-wall result S_{tw} for the action of the bounce solution in three dimensions. S_{U} and S_{L} were computed with the help of the inf-max principle (9').

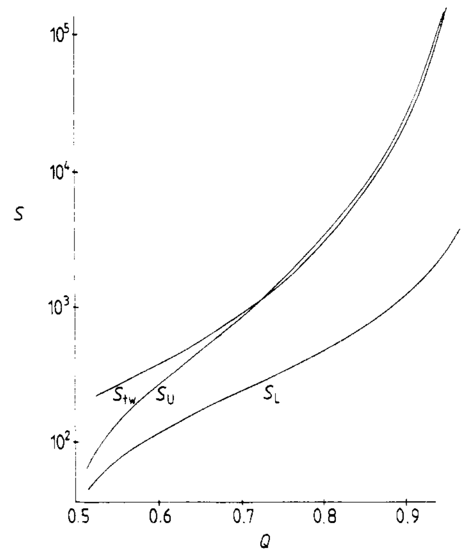

Figure 5. Plots of the lower bound S_L, upper bound S_U and thin-wall result S_{tw} for the action of the bounce solution in four dimensions. S_U is obtained with the variational principle in which the trial functions (32) and (33) were used. S_L is given in (26).

6. Concluding remarks

In QM the variational method is known to be a powerful tool for the approximate determination of the ground-state energy (see, e.g., MacDonald 1933). In this paper we generalised the Rayleigh–Ritz method to the analogue nonlinear problem of finding the action of the bounce solution. The resulting variational characterisation (9) and (9′) give us immediately lower and upper bounds for S_B. Our main results are given in (18) and (28). We applied this method to theories with potentials (4) and (5). In comparing our results with known exact ones one sees that the computed upper bounds are astonishingly good.

It is also worthwhile noting that (9) allows a short and elegant proof that ϕ_B is spherically symmetric for a large class of potentials (Wipf 1985).

We believe that (8) also holds for actions on curved spacetimes M. If M is a G manifold and S a G invariant functional then it may help to show that the bounce solution is G invariant.

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